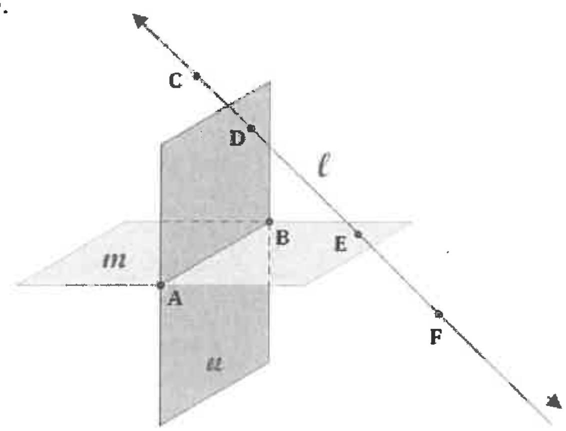
Geometry
Points, Lines & Planes

Item	Illustration	Notation	Definition
Point	•	A	A location in space.
Segment	___	$\overline{AB}$	A straight path that has two endpoints.
Ray	⟶	$\overrightarrow{AB}$	A straight path that has one endpoint and extends infinitely in one direction.
Line	⟷	ℓ or $\overleftrightarrow{AB}$	A straight path that extends infinitely in both directions.
Plane		$\boldsymbol{m}$ or ABD (points A, B, D not linear)	A flat surface that extends infinitely in two dimensions.

Collinear points are points that lie on the same line.

Coplanar points are points that lie on the same plane.



In the figure at right:

- A, B, C, D, E and F are points.
- ℓ is a line
- $\boldsymbol{m}$ and $\boldsymbol{n}$ are planes.

In addition, note that:

- C, D, E and F are **collinear points.**
- A, B and E are **coplanar points.**
- A, B and D are **coplanar points.**
- Ray $\overrightarrow{EF}$ goes off in a southeast direction.
- Ray $\overrightarrow{EC}$ goes off in a northwest direction.
- Together, rays $\overrightarrow{EF}$ and $\overrightarrow{EC}$ make up line ℓ.
- Line ℓ intersects both planes $\boldsymbol{m}$ and $\boldsymbol{n}$.

Note: In geometric figures such as the one above, it is important to remember that, even though planes are drawn with edges, they extend infinitely in the 2 dimensions shown.

An **intersection** of geometric shapes is the set of points they share in common.

ℓ and $\boldsymbol{m}$ intersect at point E.

ℓ and $\boldsymbol{n}$ intersect at point D.

$\boldsymbol{m}$ and $\boldsymbol{n}$ intersect in line $\overleftrightarrow{AB}$.

Geometry
Segments, Rays & Lines

Some Thoughts About …

Line Segments

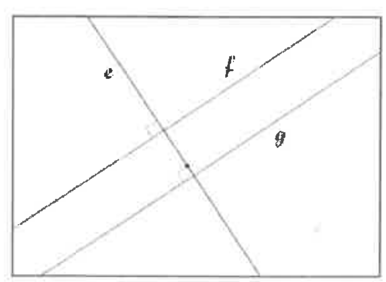
- Line segments are generally named by their endpoints, so the segment at right could be named either $\overline{AB}$ or $\overline{BA}$.
- Segment $\overline{AB}$ contains the two endpoints (A and B) and all points on line $\overleftrightarrow{AB}$ that are between them.

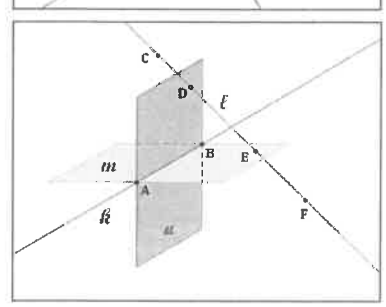
Rays

- Rays are generally named by their single endpoint, called an **initial point,** and another point on the ray.
- Ray $\overrightarrow{AB}$ contains its initial point A and all points on line $\overleftrightarrow{AB}$ in the direction of the arrow.
- Rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are not the same ray.
- If point O is on line $\overleftrightarrow{AB}$ and is between points A and B, then rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are called **opposite rays**. They have only point O in common, and together they make up line $\overleftrightarrow{AB}$.

Lines

- Lines are generally named by either a single script letter (e.g., ℓ) or by two points on the line (e.g.,. $\overleftrightarrow{AB}$).
- A line extends infinitely in the directions shown by its arrows.
- Lines are **parallel** if they are in the same plane and they never intersect. Lines f and g, at right, are parallel.
- Lines are **perpendicular** if they intersect at a 90^{0} angle. A pair of perpendicular lines is always in the same plane. Lines f and e, at right, are perpendicular. Lines g and e are also perpendicular.
- Lines are **skew** if they are not in the same plane and they never intersect. Lines k and ℓ, at right, are skew. (Remember this figure is 3-dimensional.)





Geometry
Distance Between Points

Distance measures how far apart two things are. The distance between two points can be measured in any number of dimensions, and is defined as the length of the line connecting the two points. Distance is always a positive number.

1-Dimensional Distance

In one dimension the distance between two points is determined simply by subtracting the coordinates of the points.

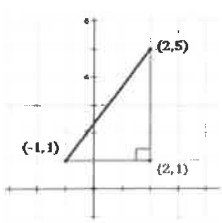
Example: In this segment, the distance between -2 and 5 is calculated as: $5-(-2)=7$.

2-Dimensional Distance

In two dimensions, the distance between two points can be calculated by considering the line between them to be the hypotenuse of a right triangle. To determine the length of this line:

- Calculate the difference in the x-coordinates of the points
- Calculate the difference in the y-coordinates of the points
- Use the Pythagorean Theorem.

This process is illustrated below, using the variable ***"d"*** for distance.



Example: Find the distance between (-1,1) and (2,5). Based on the illustration to the left:

x-coordinate difference: $2-(-1)=3$.

y-coordinate difference: $5-1=4$.

Then, the distance is calculated using the formula: $d^2=(3^2+4^2)=(9+16)=25$

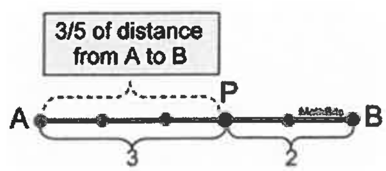
So, $\boldsymbol{d=5}$

If we define two points generally as (x_1, y_1) and (x_2, y_2), then a 2-dimensional distance formula would be:

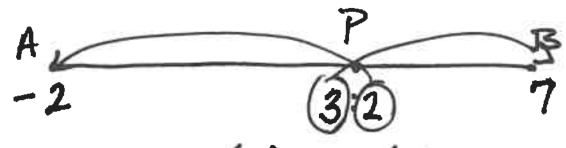
$$distance=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Partitioning a directed line segment – divide into parts.

Partitioning a line segment, AB, into a ratio a:b involves dividing the line segment into a + b equal parts. Finding a point that is "a" equal parts from point A and "b" equal parts from point B. When finding a point, P, to partition a line segment, AB, into the ratio a:b, we first find a ratio c = a / (a + b).



Eye glasses trick

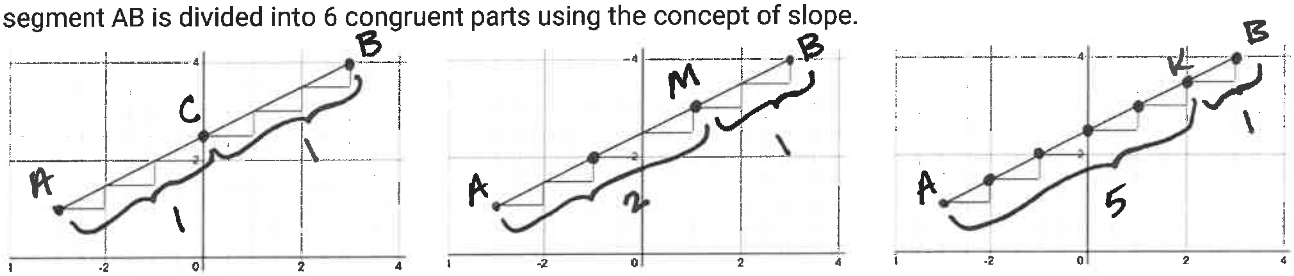


Point P has a ratio of 3:2 on segment AB.

P is 3/(3+2) = 3/5 of the way from A to B

$$P = \frac{2(-2) + 3(7)}{3+2} = \frac{-4+21}{5} = \boxed{\frac{17}{5}}$$

Partitioning a line segment, segment AB, into a ratio a:b, a/b, a to b involves dividing the line segment into a + b equal parts and finding a point that is "a" equal parts from A(-3,1) and "b" equal parts from B(3,4). Notice segment AB is divided into 6 congruent parts using the concept of slope.



Point C divides Segment AB into a ratio of 1:1 also known as the midpoint. $\frac{1}{2}$ of the segment is closest to A and the other $\frac{1}{2}$ closest to B. You could use the midpoint formula to find the coordinates of C or simply use the graph.	Point M divides segment AB into a ratio of 2:1. In this case Segment AB is be divided into 3 parts where 2 parts ($\frac{2}{3}$ of the segment) is closest to A and 1 part ($\frac{1}{3}$ of the segment) closest to B.	Point K divides segment AB into a ratio of 5:1. In this case segment AB is divided into 6 parts, 5 parts ($\frac{5}{6}$ of the segment) are closer to A and 1 part ($\frac{1}{6}$ of the segment) closest to B.

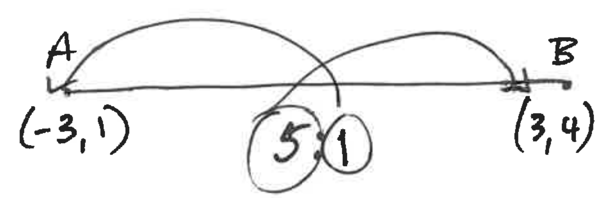
Ex using the formula

Partitioning formula

Coordinates of point which partitions a directed line segment AB at the ratio of a:b from $A(x_1, y_1)$ to $B(x_2, y_2)$

$$(x, y) = \left(\frac{bx_1 + ax_2}{b+a}, \frac{by_1 + ay_2}{b+a}\right)$$

Think of this one as the <u>**EYE GLASSES**</u> formula!



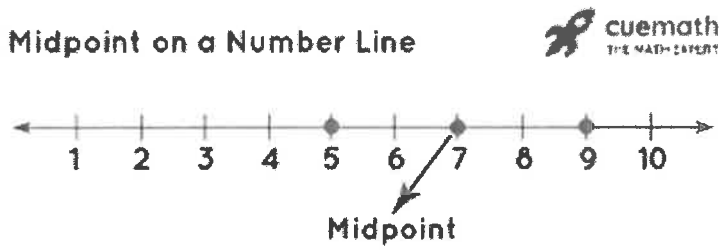
x-Coord $\frac{1(-3) + 5(3)}{5+1} = \frac{-3+15}{6} = \frac{12}{6} = 2$

y-Coord $\frac{1(1) + 5(4)}{5+1} = \frac{1+20}{6} = \frac{21}{6} = 3.5$

$\boxed{K(2, 3.5)}$

Derivation of Midpoint Formula

Let us look at this example and find the midpoint of two points in one-dimensional axis. Suppose, we have two points, 5 and 9, on a number line. The midpoint will be calculated as: (5 + 9)/2 = 14/2 = 7. So, 7 is the midpoint of 5 and 9.



Use midpoints and bisectors to find the halfway mark between two coordinates.

When two segments are congruent, we indicate that they are congruent, or of equal length, with **segment markings**, as shown below:

Figure 1.4.1

A **midpoint** is a point on a line segment that divides it into two congruent segments.

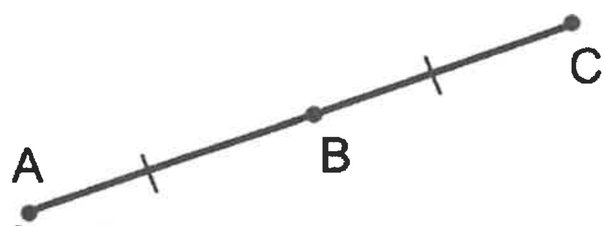
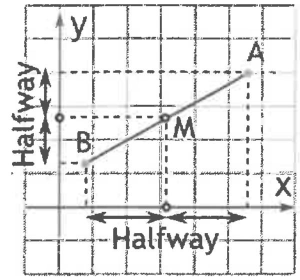


Figure 1.4.2

Midpoint of a Line Segment

The midpoint is **halfway** between the two end points:

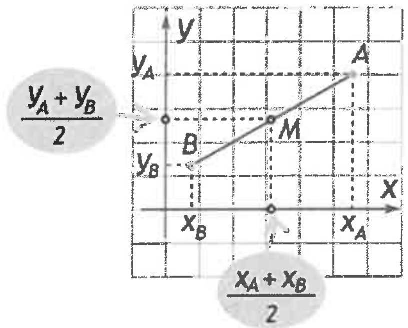
- Its **x value** is halfway between the two x values
- Its **y value** is halfway between the two y values

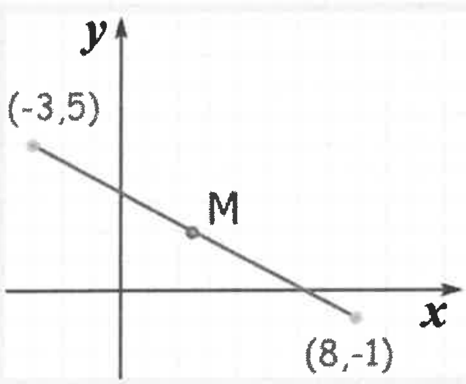


To calculate it:

- Add both "x" coordinates, divide by 2
- Add both "y" coordinates, divide by 2

In other words it's **x value** is the **average** of the x values of point A and B (and similarly for y).





Use the formula:

$$M = \left(\frac{x_A+x_B}{2} , \frac{y_A+y_B}{2} \right)$$

$$M = \left(\frac{(-3)+8}{2} , \frac{5+(-1)}{2} \right)$$

M = (5/2, 4/2)

M = (2.5, 2)

Segment Bisectors and Perpendicular Bisectors

Segment bisectors that bisect at 90° are called perpendicular bisectors. A perpendicular bisector is defined as a line or a line segment that divides a given line segment into two parts of equal measurement making four angles of 90° each on both sides. Perpendicular bisector on a line segment can be constructed easily using a ruler and a compass.

Perpendicular Bisector

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Perpendicular Bisector

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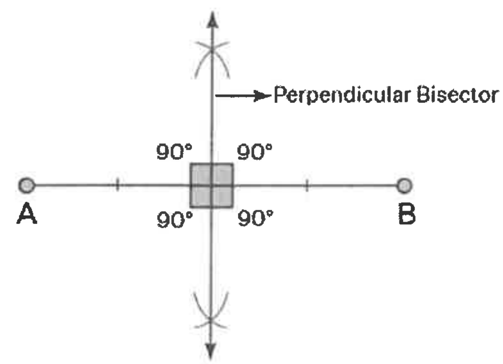


Find x and y.

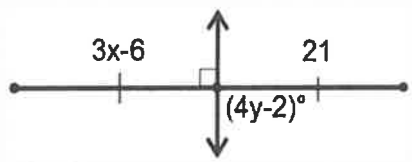


Figure 1.4.8

Solution

The line shown is the perpendicular bisector.

$$\text{So, } 3x - 6 = 21$$

$$3x = 27$$

$$x = 9$$

$$\text{And, } (4y - 2) = 90$$

$$4y = 92$$

$$y = 23$$

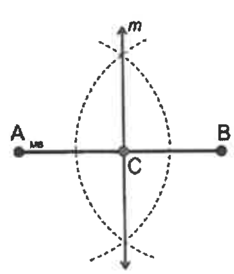
Bisect a line segment

Note: This construction is also the construction for **Perpendicular Bisector of a Segment.**

Given: $\overline{AB}$ (*a line segment*)
Construction: bisect $\overline{AB}$.

STEPS:

1. Place your compass point on *A* and stretch the compass MORE THAN half way to point *B* (you may also stretch to point *B*).
2. With this length, swing a large arc that will go above and below $\overline{AB}$.
3. Without changing the span on the compass, place the compass point on *B* and swing the arc again. The two arcs need to be extended sufficiently so they will intersect in two locations.
4. Using your straightedge, connect the two points of intersection with a line or segment to locate point *C* which bisects the segment.



$AC = CB$

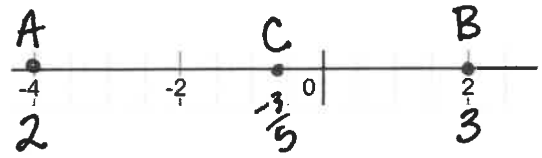
$\overline{AC} \cong \overline{CB}$

C is midpoint of $\overline{AB}$

$m \perp$ bisector $\overline{AB}$

Weighted formula

The process of finding weighted averages – place the weights on their corresponding points.
- multiply the weight and coordinate of the corresponding end point
- add those 2 products
- divide by the sum of the weights



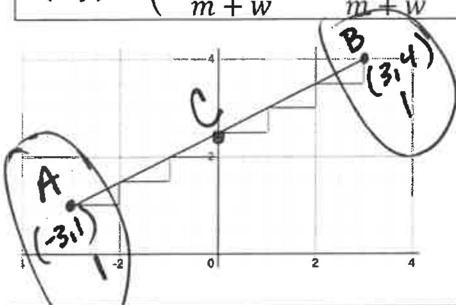
$$\frac{2(-4)+3(2)}{2+3} = \frac{-8+5}{5} = -\frac{3}{5} = C$$

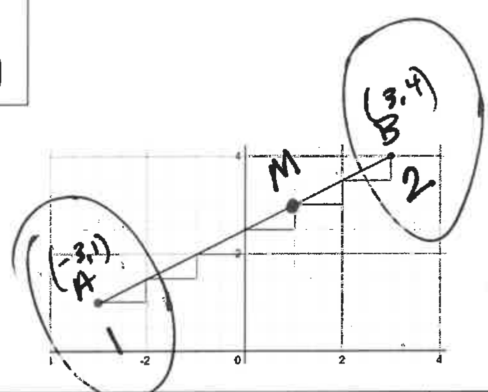
A weighs 2 and B weighs 3

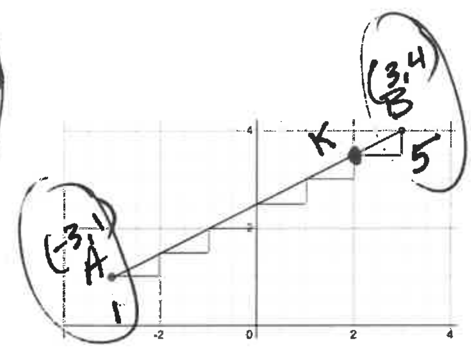
Weighted Average on a Coordinate Plane – use the above process twice, once with the x-coordinates and then again with the y-coordinates.

Weighted average (weights on a segment) can be thought of as looking for a balance point. Place the weights on the ends of the segment. A(x_1, y_1) has a weight of m and B(x_1, y_1) has a weight of w.

$$(x, y) = \left(\frac{mx_1 + wx_2}{m + w}, \frac{my_1 + wy_2}{m + w}\right)$$







Find the location of point C if point A has a weight of 1 and point B has a weight of 1. Hang the weights on the ends like on a balance.	Find the location of point M if point A has a weight of 1 and point B has a weight of 2. Hang the weights on the ends like on a balance.	Find the location of point K if point A has a weight of 1 and point B has a weight of 5. Hang the weights on the ends like on a balance.

x-coord $\frac{1(-3)+1(3)}{1+1} = \frac{0}{2} = 0$

y-coord $\frac{1(1)+1(4)}{2} = \frac{1+5}{2} = 3$

C(0,3)

x-coord $\frac{1(-3)+5(3)}{1+5} = \frac{-3+15}{6} = \frac{12}{6} = 2$

y-coord $\frac{1(1)+5(4)}{1+5} = \frac{1+20}{6} = \frac{21}{6} = 3.5$

K(2, 3.5)